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1988 J. Phys. A: Math. Gen. 21 3227

(http://iopscience.iop.org/0305-4470/21/15/011)

J. Phys. A: Math. Gen. 21 (1988) 3227-3237. Printed in the UK

Collective spin from the linearisation of $(2\lambda+1)$-dimensional Schrödinger equations as used for nuclear surface vibrations†

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Received 29 March 1988

Abstract. The free Schrödinger equation for collective multipole degrees of freedom is linearised so that energy and momentum operators appear only in first order. The wavefunction solving this linearised equation carries a collective spin, depending on multipolarity. We derive the operator of the collective spin and its eigenvalues by various methods.

1. Introduction

Spin degrees of freedom were first considered in connection with electrons. Because several experimental results, for example the doublet splitting of the alkali metals, could not be interpreted by classical means, Uhlenbeck and Goudsmit (1925, 1926) claimed the hypothesis that the electron has an intrinsic angular momentum of $\frac{1}{2}\hbar$. Shortly after that, Pauli (1927) postulated the so-called Pauli equation, which describes the spin of the electron in a non-relativistic manner. Contrary to this phenomenological approach, the spin and correct magnetic moment ($g=2$) of the electron were directly derived by Dirac (1928) from his relativistic equation. In this sense it might be obvious to mark the spin and spin magnetic moment as quantities generated by the theory of relativity, but this seems not to be true. Inspired by the fact that the free Dirac equation had been constructed from a linearisation of the free Klein-Gordon equation, Lévy-Leblond (1967) showed that the Pauli equation is obtained if the free Schrödinger equation is linearised, i.e. an equivalent equation with energy and momentum appearing only in first order is constructed, and electromagnetic potentials are coupled minimally. Therefore, we can assume that the spin and spin magnetic moment of the electron are not relativistic quantities, but originate due to the linearisation of the equation of motion.

In this paper we want to apply the same linearisation procedure as known from the case of electrons to physical systems which are described with multipole degrees of freedom. Multipole degrees of freedom are related to the collective dynamics of physical systems. Therefore, the linearisation of Schrödinger equations of collective multipole degrees of freedom leads to collective spin degrees of freedom in complete analogy to the case of the Dirac and Pauli equations.

† Work supported by BMFT and GSI (Darmstadt).

0305-4470/88/153227+11$02.50

For example, in nuclear physics, collective properties of the nucleus, as surface vibrations of a spherical nucleus, rotational motion of a deformed nucleus or giant resonances, can be described by collective multipole coordinates (Eisenberg and Greiner 1987, Bohr and Mottelson 1975). As shown in a short note (Greiner *et al* 1988), the linearisation of the Schrödinger equation for collective nuclear quadrupole surface vibrations yields a collective spin of $\frac{3}{2}\hbar$. Also collective spins for higher multipolarities have been derived in this note.

The aim of this paper is to show different derivations of the collective spin from the linearised free Schrödinger equations for collective multipole degrees of freedom. In § 2 we briefly sketch the linearisation procedure of a multi-dimensional free Schrödinger equation. Section 3 is devoted to the various methods of deducing the collective spin of any multipolarity. Results are presented in § 4. In § 5 we then try to indicate possible applications of the concept of collective spin.

2. Linearisation of the free $(2\lambda+1)$-dimensional Schrödinger equation

Schrödinger equations describing $2\lambda+1$ multipole degrees of freedom are used for example for the treatment of nuclear collective surface vibrations. In this case the vibrations of spherical nuclei are described by collective surface coordinates $\alpha_{\lambda\mu}$ defined by the expansion of the nuclear surface as follows (Eisenberg and Greiner 1987)

$$R(\theta,\phi)=R_0\left[1+\sum_{\lambda,\mu}(-1)^{\mu}\alpha_{\lambda-\mu}Y_{\lambda\mu}(\theta,\phi)\right]. \tag{1}$$

The Hamiltonian for surface vibrations of multipolarity λ is rotationally invariant and has the following structure in lowest order in these coordinates:

$$H_\lambda=(-1)^{\lambda}\frac{(2\lambda+1)^{1/2}}{2B_\lambda}[\pi^{[\lambda]}\otimes\pi^{[\lambda]}]^{[0]}+(-1)^{\lambda}\frac{(2\lambda+1)^{1/2}C_\lambda}{2}[\alpha^{[\lambda]}\otimes\alpha^{[\lambda]}]^{[0]} \tag{2}$$

$$=\frac{1}{2B_\lambda}\sum_{\mu}(-1)^{\mu}\pi_{\lambda\mu}\pi_{\lambda-\mu}+\frac{C_\lambda}{2}\sum_{\mu}(-1)^{\mu}\alpha_{\lambda\mu}\alpha_{\lambda-\mu}.$$

The quantities $\pi_{\lambda\mu}$ are the canonically conjugate momenta. Since the coordinates $\alpha_{\lambda\mu}$ are complex,

$$\alpha^{*}_{\lambda\mu}=(-1)^{\mu}\alpha_{\lambda-\mu}$$

we first introduce real coordinates $x_i^{(\lambda)}$, which are easier to handle:

$$\begin{aligned}
x^{(\lambda)}_{\lambda+1-\mu}&=\frac{1}{\sqrt{2}}[\alpha_{\lambda\mu}+(-1)^{\mu}\alpha_{\lambda-\mu}]\\
x^{(\lambda)}_{\lambda+1}&=\alpha_{\lambda 0}\\
x^{(\lambda)}_{\lambda+1+\mu}&=\frac{\mathrm{i}}{\sqrt{2}}[\alpha_{\lambda\mu}-(-1)^{\mu}\alpha_{\lambda-\mu}]
\end{aligned} \tag{3a}$$

with $\mu=1,\ldots,\lambda$.

In shorthand notation these transformations read

$$x_i^{(\lambda)}=\sum_j \mathbb{X}_{ij}^{(\lambda)}\alpha_{\lambda j}$$

where the transformation matrix $\mathbb{X}^{(\lambda)}$ is defined by equation (3*a*). The corresponding momenta are given by ($\mu = 1, \ldots, \lambda$)

$$\begin{aligned} p^{(\lambda)}_{\lambda+1-\mu} &= \frac{1}{\sqrt{2}}[\pi_{\lambda\mu} + (-1)^{\mu}\pi_{\lambda-\mu}] \\ p^{(\lambda)}_{\lambda+1} &= \pi_{\lambda 0} \\ p^{(\lambda)}_{\lambda+1+\mu} &= \frac{-\mathrm{i}}{\sqrt{2}}[\pi_{\lambda\mu} - (-1)^{\mu}\pi_{\lambda-\mu}] \end{aligned} \tag{3b}$$

or in shorthand notation

$$p^{(\lambda)}_i = \sum_j \mathbb{P}^{(\lambda)}_{ij}\pi_{\lambda j}.$$

The new coordinates and momenta fulfil the usual commutation relations

$$[p^{(\lambda)}_i, x^{(\lambda)}_j] = -\mathrm{i}\hbar\delta_{ij}.$$

Using these coordinates and momenta we can rewrite the Hamiltonian (2) as follows:

$$H_\lambda = \frac{1}{2B_\lambda}\sum_{i=1}^{2\lambda+1} p^{(\lambda)2}_i + \frac{C_\lambda}{2}\sum_{i=1}^{2\lambda+1} x^{(\lambda)2}_i. \tag{4}$$

This form is more adequate for the linearisation procedure than (2) in the spherical representation.

In a completely analogous manner as that in which the three-dimensional Schrödinger equation is linearised (Lévy-Leblond 1967) or the Dirac equation is derived from the Klein-Gordon equation (Dirac 1928), we linearise the $(2\lambda+1)$-dimensional free Schrödinger equation

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \frac{1}{2B_\lambda}\sum_{i=1}^{2\lambda+1} p^{(\lambda)2}_i\right)\psi = 0. \tag{5}$$

The linearised free Schrödinger equation must have the following structure:

$$\theta_\lambda\psi := \left[P^{(\lambda)}\left(\mathrm{i}\hbar\frac{\partial}{\partial t}\right) + \sum_{i=1}^{2\lambda+1} Q^{(\lambda)}_i p^{(\lambda)}_i + R^{(\lambda)}\right]\psi = 0. \tag{6}$$

In this equation the time derivative and momenta appear only in first order. The operators $P^{(\lambda)}$, $Q^{(\lambda)}_i$ and $R^{(\lambda)}$ will be determined next. We assume that the wavefunction ψ solves the free Schrödinger equation (5) and the linearised free Schrödinger equation (6). This is achieved by the definition of a second linear operator

$$\theta'_\lambda := P^{(\lambda)'}\left(\mathrm{i}\hbar\frac{\partial}{\partial t}\right) + \sum_{i=1}^{2\lambda+1} Q^{(\lambda)'}_i p^{(\lambda)}_i + R^{(\lambda)'} \tag{7}$$

with the property

$$\theta'_\lambda\theta_\lambda = 2B_\lambda\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \frac{1}{2B_\lambda}\sum_{i=1}^{2\lambda+1} p^{(\lambda)2}_i\right). \tag{8}$$

Using the abbreviations

$$Q^{(\lambda)}_{2\lambda+2} = \mathrm{i}\left(P^{(\lambda)} + \frac{1}{2B_\lambda}R^{(\lambda)}\right)$$

$$Q_{2\lambda+2}^{(\lambda)'} = \mathrm{i}\left(P^{(\lambda)'} + \frac{1}{2B_\lambda} R^{(\lambda)'}\right)$$
$$Q_{2\lambda+3}^{(\lambda)} = P^{(\lambda)} - \frac{1}{2B_\lambda} R^{(\lambda)} \tag{9}$$
$$Q_{2\lambda+3}^{(\lambda)'} = P^{(\lambda)'} - \frac{1}{2B_\lambda} R^{(\lambda)'}$$

we obtain from (8) the conditions

$$Q_i^{(\lambda)'} Q_j^{(\lambda)} + Q_j^{(\lambda)'} Q_i^{(\lambda)} = -2\delta_{ij} I \qquad i,j = 1,\ldots,2\lambda+3. \tag{10}$$

If we choose

$$Q_i^{(\lambda)} = M^{(\lambda)} \gamma_i^{(\lambda)} \qquad i = 1,\ldots,2\lambda+2$$
$$Q_{2\lambda+3}^{(\lambda)} = -iM^{(\lambda)} \tag{11a}$$

where $M^{(\lambda)}$ is an arbitrary, but non-singular matrix, the primed Q-operators have to be

$$Q_i^{(\lambda)'} = -\gamma_i^{(\lambda)} M^{(\lambda)-1} \qquad i = 1,\ldots,2\lambda+2$$
$$Q_{2\lambda+3}^{(\lambda)'} = -\mathrm{i} M^{(\lambda)-1}. \tag{11b}$$

Inserting (11a) and (11b) into (10) we find that the $2\lambda+2$ matrices $\gamma_i^{(\lambda)}$ ($i = 1,\ldots,2\lambda+2$) have to fulfil the Clifford algebra:

$$\gamma_i^{(\lambda)} \gamma_j^{(\lambda)} + \gamma_j^{(\lambda)} \gamma_i^{(\lambda)} = 2\delta_{ij} I \qquad i,j = 1,\ldots,2\lambda+2. \tag{12}$$

An irreducible representation of these matrices is given by the following scheme: let $\gamma_i^{(1)}$ be defined by the usual 4×4 Dirac matrices

$$\gamma_i^{(1)} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \text{for } i = 1,2,3$$
$$\gamma_4^{(1)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{13}$$

Here, the σ_i matrices are the well known Pauli matrices. Then the $2\lambda+2$ matrices $\gamma_i^{(\lambda)}$ can be successively calculated using the relations

$$\gamma_i^{(\lambda)} = \begin{pmatrix} 0 & \gamma_i^{(\lambda-1)} \\ \gamma_i^{(\lambda-1)} & 0 \end{pmatrix} \qquad \text{for } i = 1,\ldots,2\lambda+1 \tag{14a}$$

$$\gamma_{2\lambda+2}^{(\lambda)} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{14b}$$

with

$$\gamma_{2\lambda+1}^{(\lambda-1)} = \mathrm{i}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{14c}$$

The scheme (14) allows one to construct the γ matrices of multipolarity λ from those of multipolarity $\lambda-1$.

For simplicity we set $M^{(\lambda)} = I$ in the following. Then the linearised $(2\lambda+1)$-dimensional free Schrödinger equation (6) finally reads

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \left(\mathrm{i}\hbar \frac{\partial}{\partial t} \right) \psi = H_{\mathrm{L}}^{(\lambda)} \psi \tag{15}$$

$$H_{\mathrm{L}}^{(\lambda)} = -\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} 2B_\lambda - \sum_{i=1}^{2\lambda+1} \begin{pmatrix} 0 & \mathrm{i}\gamma_i^{(\lambda-1)} \\ -\mathrm{i}\gamma_i^{(\lambda-1)} & 0 \end{pmatrix} p_i^{(\lambda)}. \tag{16}$$

The wavefunction ψ can be written as

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Both functions ϕ and χ satisfy the free Schrödinger equation (5).

3. Deduction of the collective spin

The linearised free Schrödinger equation (15) contains spin degrees of freedom, which we denote as collective spin degrees of freedom. This notation is justified since Schrödinger equations for multipole degrees of freedom generally describe collective degrees of freedom, e.g. the nuclear surface vibrations.

In the following we present three methods for the deduction of the collective spin. The first one is an investigation of the commutator between the angular momentum operator and the linearised free Hamiltonian from which the collective spin results. The transformation property of the wavefunction under spatial rotations is the subject of the second method and leads also to the same collective spin. In the third approach the time reversal operator depending on the collective spin will be constructed.

3.1. Analogy between angular momentum and spin tensor

The angular momentum operator of multipolarity λ is, in spherical components, (Eisenberg and Greiner 1987)

$$L_{1\mu}^{(\lambda)} = -M_\lambda \sum_{\sigma,\nu} (\lambda\lambda 1|\sigma\nu\mu) \alpha_{\lambda\sigma} \pi_{\lambda\nu}^{*}. \tag{17}$$

The constant factor M_λ can be determined from the requirement that the angular momentum commutator algebra for the Cartesian components, $[L_i, L_j] = \mathrm{i}\hbar\varepsilon_{ijk}L_k$ with $i, j, k = 1, 2, 3$, is fulfilled:

$$M_\lambda = \mathrm{i}(-1)^\lambda [\tfrac{1}{3}\lambda(\lambda+1)(2\lambda+1)]^{1/2}. \tag{18}$$

If the Cartesian components of the angular momentum operator are expressed within the new coordinates $x_i^{(\lambda)}$ and momenta $p_i^{(\lambda)}$, they will be specific linear combinations of elements of the angular momentum tensor $L_{mn}^{(\lambda)}$

$$L_i^{(\lambda)} = \mathscr{L}_i^{(\lambda)}(L_{mn}^{(\lambda)}) \qquad \text{for } i = 1, 2, 3 \tag{19}$$

where

$$L_{mn}^{(\lambda)} = x_m^{(\lambda)} p_n^{(\lambda)} - x_n^{(\lambda)} p_m^{(\lambda)}. \tag{20}$$

The symbol $\mathscr{L}$ stands for linear combination. For example, for quadrupole degrees of freedom the angular momentum operators are

$$\begin{aligned} L_x^{(2)} &= L_{41}^{(2)} + L_{52}^{(2)} + \sqrt{3} L_{43}^{(2)} \\ L_y^{(2)} &= L_{12}^{(2)} + L_{54}^{(2)} + \sqrt{3} L_{23}^{(2)} \\ L_z^{(2)} &= 2L_{51}^{(2)} + L_{42}^{(2)}. \end{aligned} \tag{21}$$

Generally the z-component can be written as

$$L_z^{(\lambda)} = \sum_{\mu=1}^{\lambda} (\lambda + 1 - \mu) L_{2\lambda+2-\mu,\mu}^{(\lambda)}. \tag{22}$$

Each element of the angular momentum tensor $L_{mn}^{(\lambda)}$ is of course Hermitian. The following commutator algebra between two elements is valid:

$$[L_{ij}^{(\lambda)}, L_{mn}^{(\lambda)}] = -\mathrm{i}\hbar(\delta_{jm} L_{in}^{(\lambda)} + \delta_{jn} L_{mi}^{(\lambda)} + \delta_{im} L_{nj}^{(\lambda)} + \delta_{in} L_{jm}^{(\lambda)}). \tag{23}$$

The commutator between the free linearised Hamiltonian $H_{\mathrm{L}}^{(\lambda)}$ of (16) and $L_{mn}^{(\lambda)}$ yields

$$[H_{\mathrm{L}}^{(\lambda)}, L_{mn}^{(\lambda)}] = \mathrm{i}\hbar \left[\begin{pmatrix} 0 & \mathrm{i}\gamma_m^{(\lambda-1)} \\ -\mathrm{i}\gamma_m^{(\lambda-1)} & 0 \end{pmatrix} p_n^{(\lambda)} - \begin{pmatrix} 0 & \mathrm{i}\gamma_n^{(\lambda-1)} \\ -\mathrm{i}\gamma_n^{(\lambda-1)} & 0 \end{pmatrix} p_m^{(\lambda)} \right]. \tag{24}$$

Because the right-hand side of this equation does not vanish, the commutator between $H_{\mathrm{L}}^{(\lambda)}$ and the Cartesian components (19) of the angular momentum operator also does not vanish.

Introducing a Hermitian spin tensor

$$S_{mn}^{(\lambda)} = -\tfrac{1}{4}\mathrm{i}\hbar[\gamma_m^{(\lambda)}, \gamma_n^{(\lambda)}] \tag{25}$$

we find

$$[H_{\mathrm{L}}^{(\lambda)}, L_{mn}^{(\lambda)} + S_{mn}^{(\lambda)}] = 0. \tag{26}$$

In addition the commutator between two elements of the spin tensor yields the same algebra as in the case of the orbital angular momentum tensor (see (23))

$$[S_{ij}^{(\lambda)}, S_{mn}^{(\lambda)}] = -\mathrm{i}\hbar(\delta_{jm} S_{in}^{(\lambda)} + \delta_{jn} S_{mi}^{(\lambda)} + \delta_{im} S_{nj}^{(\lambda)} + \delta_{in} S_{jm}^{(\lambda)}). \tag{27}$$

As a consequence one can immediately define the Cartesian components of the spin operator analogously to the angular momentum operator given in (19):

$$S_i^{(\lambda)} = \mathscr{L}_i^{(\lambda)}(S_{mn}^{(\lambda)}) \qquad \text{for } i = 1, 2, 3. \tag{28}$$

For quadrupole degrees of freedom the spin operators are then

$$\begin{aligned} S_x^{(2)} &= S_{41}^{(2)} + S_{52}^{(2)} + \sqrt{3} S_{43}^{(2)} \\ S_y^{(2)} &= S_{12}^{(2)} + S_{54}^{(2)} + \sqrt{3} S_{23}^{(2)} \\ S_z^{(2)} &= 2S_{51}^{(2)} + S_{42}^{(2)} \end{aligned} \tag{29}$$

or in explicit irreducible form

$$S_i^{(2)} = \begin{pmatrix} \tilde{S}_i^{(2)} & 0 \\ 0 & \tilde{S}_i^{(2)} \end{pmatrix}$$

with

$$\tilde{S}_x^{(2)} = \tfrac{1}{2}\hbar \begin{pmatrix} \sigma_y & -\mathrm{i}(\sigma_x + \sqrt{3}\sigma_z) \\ \mathrm{i}(\sigma_x + \sqrt{3}\sigma_z) & -\sigma_y \end{pmatrix}$$

$$\tilde{S}_y^{(2)} = \tfrac{1}{2}\hbar \begin{pmatrix} \sigma_z + \sqrt{3}\sigma_x & -I \\ -I & \sigma_z + \sqrt{3}\sigma_x \end{pmatrix} \tag{30}$$

$$\tilde{S}_z^{(2)} = \tfrac{1}{2}\hbar \begin{pmatrix} 2\sigma_x & -\mathrm{i}\sigma_y \\ \mathrm{i}\sigma_y & -2\sigma_x \end{pmatrix}.$$

For arbitrary multipolarity λ the z-component of the spin operator is equal to

$$S_z^{(\lambda)} = \sum_{\mu=1}^{\lambda} (\lambda + 1 - \mu) S_{2\lambda+2-\mu,\,\mu}^{(\lambda)}. \tag{31}$$

In conclusion, we have derived the collective spin operator for any multipolarity λ using the analogy between the orbital angular momentum tensor and the spin tensor.

3.2. Transformation property of the wavefunction under spatial rotations

For a clear presentation we will restrict this approach to quadrupole degrees of freedom only, e.g. to quadrupole surface vibrations. In principle it can be handled for any multipolarity analogously to the quadrupole case, but the treatment will become rather tedious.

The total angular momentum operator of a system is determined by the transformation property of the wavefunction under spatial rotations

$$\psi' = \exp(\mathrm{i}/\hbar \boldsymbol{\phi} \boldsymbol{J})\psi = \exp[\mathrm{i}/\hbar \boldsymbol{\phi}(\boldsymbol{L} + \boldsymbol{S})]\psi \tag{32}$$

where ψ' is the wavefunction in the rotated coordinate system and ψ is the wavefunction in the fixed one; $\boldsymbol{\phi}$ represents the angle of rotation between the two coordinate systems. Note the plus sign in the exponent of (32) which indicates that the coordinate system is rotated. The wavefunction ψ is an eight-component spinor in the quadrupole case. Its transformation behaviour can be specified, for example, with the Lagrangian density corresponding to the linearised free Schrödinger equation (15):

$$\mathscr{L}^{(2)}(x_i^{(2)}) = \psi^{+}(x_i^{(2)}) \left[\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \mathrm{i}\hbar \frac{\partial}{\partial t} - H_{\mathrm{L}}^{(2)}(x_i^{(2)}) \right] \psi(x_i^{(2)}). \tag{33}$$

Then the invariance of the scalar quantity

$$\mathscr{L}^{(2)'}(x_i^{(2)'}) = \mathscr{L}^{(2)}(x_i^{(2)}) \tag{34}$$

completely determines the transformation property of the wavefunction under spatial rotations. It is sufficient to consider infinitesimal rotations only.

In order to extract the correct transformation properties of the wavefunction under spatial rotations from the requirement (34), let us first consider the transformation behaviour of the coordinates and momenta. The following relations hold for the spherical quadrupole coordinates $\alpha_{2\mu}$ and momenta $\pi_{2\mu}$ (Eisenberg and Greiner 1987)

$$\alpha'_{2\mu} = \sum_{\nu} D^{2}_{\nu\mu}(\alpha, \beta, \gamma)\alpha_{2\nu} \tag{35a}$$

$$\pi'_{2\mu} = \sum_{\nu} D^{2*}_{\nu\mu}(\alpha, \beta, \gamma)\pi_{2\nu}. \tag{35b}$$

The matrix D^2 (for definition see Rose (1957)) depends on the three Euler angles α, β and γ, defined as follows: α describes a rotation around the z-axis, β a rotation around the new y-axis and γ a rotation around the z'-axis. With that the following result yields for $x_i^{(2)'}$ and $p_i^{(2)'}$:

$$x_i^{(2)'} = \sum_{j,m,n} \mathbb{X}_{im}^{(2)} D_{nm}^{2}(\alpha, \beta, \gamma) \mathbb{X}_{jn}^{(2)*} x_j^{(2)}$$
$$= \sum_j R_{ij}^{(2)}(\alpha, \beta, \gamma) x_j^{(2)} \tag{36a}$$

$$p_i^{(2)'} = \sum_{j,m,n} \mathbb{P}_{im}^{(2)} D_{nm}^{2*}(\alpha, \beta, \gamma) \mathbb{P}_{jn}^{(2)*} p_j^{(2)}$$
$$= \sum_j R_{ij}^{(2)}(\alpha, \beta, \gamma) p_j^{(2)}. \tag{36b}$$

Since the coordinates $x_i^{(2)}$ are real, the matrix $R^{(2)}$ is an orthogonal matrix. For infinitesimal rotations we obtain

$$(R_{ij}^{(2)}(\delta\alpha, \delta\beta, \delta\gamma)) = \begin{pmatrix} 1 & \delta\beta & 0 & 0 & -2(\delta\alpha+\delta\gamma) \\ -\delta\beta & 1 & \sqrt{3}\delta\beta & -(\delta\alpha+\delta\gamma) & 0 \\ 0 & -\sqrt{3}\delta\beta & 1 & 0 & 0 \\ 0 & \delta\alpha+\delta\gamma & 0 & 1 & -\delta\beta \\ 2(\delta\alpha+\delta\gamma) & 0 & 0 & \delta\beta & 1 \end{pmatrix}. \tag{37}$$

For these infinitesimal rotations equation (32) now becomes

$$\psi'(x_i^{(2)'}) = \left(1 + \frac{\mathrm{i}}{\hbar}[\delta\beta(L_y^{(2)} + S_y^{(2)}) + (\delta\alpha+\delta\gamma)(L_z^{(2)} + S_z^{(2)})]\right)\psi(x_i^{(2)'})$$
$$= \left(1 + \frac{\mathrm{i}}{\hbar}[\delta\beta S_y^{(2)} + (\delta\alpha+\delta\gamma) S_z^{(2)}]\right)\psi(x_i^{(2)}). \tag{38}$$

From the first step of this transformation it becomes obvious that no rotations about the x-axis can be generated with infinitesimal Euler angles; equations (21), (36) and (37) were used in the second step.

It remains to clarify the transformation properties of the γ matrices. The $\gamma_i^{(2)'}$ matrices of the rotated coordinate system have to fulfil the Clifford algebra (12) and have to be Hermitian of course. Because of these two conditions the primed $\gamma_i^{(2)'}$ matrices are given by the corresponding $\gamma_i^{(2)}$ matrices of the fixed coordinate system by means of a unitary transformation $U(\alpha, \beta, \gamma)$ (Good 1955)

$$\gamma_i^{(2)'} = U \gamma_i^{(2)} U^{-1}. \tag{39}$$

A unitary transformation does not change the physics, so that as a consequence $\gamma_i^{(2)'}$ can be set equal to $\gamma_i^{(2)}$

$$\gamma_i^{(2)'} = \gamma_i^{(2)} \qquad i = 1, \ldots, 6. \tag{40}$$

With the equations (14), (36)-(38) and (40), the requirement (34) of rotational invariance of the Lagrangian density finally yields

$$S_i^{(2)} = \begin{pmatrix} \tilde{S}_i^{(2)} & 0 \\ 0 & \tilde{S}_i^{(2)} \end{pmatrix}$$
$$[\tilde{S}_y^{(2)}, \gamma_1^{(1)}] = \mathrm{i}\hbar\gamma_2^{(1)}$$
$$[\tilde{S}_y^{(2)}, \gamma_2^{(1)}] = \mathrm{i}\hbar(\sqrt{3}\gamma_3^{(1)} - \gamma_1^{(1)})$$
$$[\tilde{S}_y^{(2)}, \gamma_3^{(1)}] = -\mathrm{i}\hbar\sqrt{3}\gamma_2^{(1)}$$

$$
\begin{aligned}
&[\tilde{S}_y^{(2)}, \gamma_4^{(1)}] = -\mathrm{i}\hbar\gamma_5^{(1)} \\
&[\tilde{S}_y^{(2)}, \gamma_5^{(1)}] = \mathrm{i}\hbar\gamma_4^{(1)} \\
&[\tilde{S}_z^{(2)}, \gamma_1^{(1)}] = -2\mathrm{i}\hbar\gamma_5^{(1)} \\
&[\tilde{S}_z^{(2)}, \gamma_2^{(1)}] = -\mathrm{i}\hbar\gamma_4^{(1)} \\
&[\tilde{S}_z^{(2)}, \gamma_3^{(1)}] = 0 \\
&[\tilde{S}_z^{(2)}, \gamma_4^{(1)}] = \mathrm{i}\hbar\gamma_2^{(1)} \\
&[\tilde{S}_z^{(2)}, \gamma_5^{(1)}] = 2\mathrm{i}\hbar\gamma_1^{(1)}.
\end{aligned} \tag{41}
$$

The spin operators fulfilling these commutator relations are given by (29) and (30) in agreement with the results of § 3.1.

3.3. Time reversal symmetry

For isolated physical systems the symmetry under time reversal is assumed to be valid. This means that the physics does not change when the progression in time is reversed. For spinless systems the corresponding time reversal operator is simply the complex conjugation operator K (see, e.g., Schiff 1968). Because the introduced coordinates $x_i^{(\lambda)}$ are real and the following relation holds for the spherical momenta (Eisenberg and Greiner 1987),

$$K\pi_{\lambda\mu}K^{-1} = \pi_{\lambda\mu}^* = -(-1)^{\mu}\pi_{\lambda-\mu}$$

the following equations are rather obvious

$$
\begin{aligned}
&Kx_i^{(\lambda)}K^{-1} = x_i^{(\lambda)*} = x_i^{(\lambda)} \\
&Kp_i^{(\lambda)}K^{-1} = p_i^{(\lambda)*} = -p_i^{(\lambda)} \\
&KL_i^{(\lambda)}K^{-1} = L_i^{(\lambda)*} = -L_i^{(\lambda)}.
\end{aligned} \tag{42}
$$

Since the spin operator is also an angular momentum operator, it has the same behaviour under time reversal as the latter operator. Introducing a new time reversal operator T for systems with spin, we have to require the relation

$$
\begin{aligned}
T^{(\lambda)}S_i^{(\lambda)}T^{(\lambda)-1} &= T_0^{(\lambda)}KS_i^{(\lambda)}K^{-1}T_0^{(\lambda)-1} \\
&= T_0^{(\lambda)}S_i^{(\lambda)*}T_0^{(\lambda)-1} \\
&= -S_i^{(\lambda)}.
\end{aligned} \tag{43}
$$

$T_0^{(\lambda)}$ is an additional operator which only acts on the spin degrees of freedom.

In order to derive an explicit expression for $T_0^{(\lambda)}$, we assume that the linearised Schrödinger equation (15) is time reversal invariant. In other words, $H_{\mathrm{L}}^{(\lambda)}$ has to commute with $T^{(\lambda)}$. In the following we treat again only the quadrupole case.

The requirement of invariance

$$[T^{(2)}, H_{\mathrm{L}}^{(2)}] = 0 \tag{44}$$

leads to

$$\tilde{T}_0^{(2)}\gamma_i^{(1)*}\tilde{T}_0^{(2)-1} = \gamma_i^{(1)} \tag{45a}$$

where $\tilde{T}_0^{(2)}$ is defined via the matrix

$$T_0^{(2)} = \begin{pmatrix} \tilde{T}_0^{(2)} & 0 \\ 0 & \tilde{T}_0^{(2)} \end{pmatrix}. \tag{45b}$$

The solution of (45*a*) is

$$T_0^{(2)} = \exp[-i\pi/\hbar(S_{41}^{(2)} + S_{52}^{(2)} + \sqrt{3}S_{43}^{(2)})]$$
$$= \exp[-i\pi/\hbar S_x^{(2)}]. \tag{46}$$

This result assures the validity of (43).

Let us briefly summarise the result. The requirement of time reversal symmetry leads to an expression for the x-component of the spin operator for quadrupole degrees of freedom in accordance with the results of §§ 3.1 and 3.2.

4. Results for collective spin eigenvalues

The spins as functions of the multipolarity λ of the coordinates can be calculated by diagonalising the corresponding irreducible spin operators. The results are shown for $\lambda \leqslant 7$ in table 1. They can also be derived by decomposing the half-integer group representation of SO($2\lambda+1$) to SO(3) and are given in table form by Armstrong and Judd (1970). The spin values are $\frac{3}{2}$ for the quadrupole case, 0 and 3 for the octupole case and 2 and 5 for the hexadecupole case. They depend on multipolarity because the number of collective coordinates and, therefore, the number of required γ matrices increases. The number of γ matrices of multipolarity λ is $2\lambda+2$ and their dimension $2^{\lambda+1}$. For multipolarities $\lambda \geqslant 3$ several spin multiplets appear. The largest spin value of each multipolarity λ is obtained as $s_{\max} = \lambda(\lambda+1)/4$.

Table 1. The collective spin s for different multipolarities λ.

λ						
1	2	3	4	5	6	7
$\frac{1}{2}$	$\frac{3}{2}$	3	5	$\frac{15}{2}$	$\frac{21}{2}$	14
		0	2	$\frac{9}{2}$	$\frac{15}{2}$	11
				$\frac{5}{2}$	$\frac{11}{2}$	9
					$\frac{9}{2}$	8
					$\frac{3}{2}$	7
						5
						4
						2

5. Concluding remarks

The linearisation of the collective Schrödinger equation of nuclear surface vibrations, that means the construction of an equivalent equation with energy and momentum appearing only in first order, yields a new spin degree of freedom and, therefore, describes new physics. Whereas the usual collective Schrödinger equation describes a spinless system, i.e. surface vibrations of an even-even nucleus, the linearised collective Schrödinger equation describes a system with spin degrees of freedom depending on the multipolarity of the surface vibrations. The situation is similar to the transition from the Klein-Gordon equation to the Dirac equation. The Klein-Gordon equation describes bosons with spin 0 (e.g., pions), whereas the Dirac equation describes fermions with spin $\frac{1}{2}$ (e.g., electrons).

The question arises where these collective spin modes are realised in nature. A first application of the collective spin to nuclear quadrupole degrees of freedom has been considered by Wu *et al* (1988). In their fermion dynamical symmetry model (Wu *et al* 1986) which is related to the interacting boson model (Arima and Iachello 1981) it is assumed that the active pseudo-spin of a nucleon must be $\frac{3}{2}$ (*i*-active scheme) in order to produce angular momenta 0 and 2 if two identical nucleons are coupled to a pair. They have shown that the collective spin $\frac{3}{2}$ is also related to the U(6/4) nuclear supersymmetry (Wu *et al* 1988).

Another, at first more mathematical, application would be the introduction of a scalar potential or a vector potential in the linearised collective Schrödinger equation. For nuclear quadrupole surface degrees of freedom this would lead to a model of a linearised vibrator or a linearised rotator or even to a linearised rotation-vibration model if the equations of motion are transformed to a rotating coordinate system. One can investigate which aspects of even-odd nuclei are describable with these models. One may also think of a linearised treatment of the coupling between several surface degrees of freedom of different multipolarity such as quadrupole and octupole ones. Also one can examine linearised isovectorial oscillations, such as the oscillation of a proton liquid against a neutron liquid both with the intrinsic spin $\frac{3}{2}$ of the quadrupole case.

The proposed concept of linearisation, which is shown for nuclear surface vibrations in this publication, could be directly transferred to other nuclear collective degrees of freedom. For example, the collective density vibrations in the case of giant resonances could be treated in the linearised picture as well as their interaction with surface vibrations. This would lead to generalised linearised collective models. Such models may have applications also in other fields of physics, e.g. in elementary particle physics.

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